JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 45, 210–233 (1974)

Convex Games and Extreme Points

FREDDY DELBAEN

Department of Mathematics, Vrije Universiteit Brussel, 1050 Brussel, Belgium

Submitted by Ky Fan

INTRODUCTION AND ACKNOWLEDGMENTS

The theory of games in characteristic function form is well developed. However not so much work has been done on the continuity properties of the core of these games. The aim of this paper is to prove that for convex games the core behaves nicely. (We also present some counterexamples for non convex games.) In Section 1 we give some definitions and notations and prove the fundamental lemma of our paper. Section 2 lists different characterizations of convex games. In Section 3 we prove the so-called projection theorem. Roughly this theorem states that the set of core restrictions is the core of the subgame. In Section 4 we give a characterization of the extreme points of the core of a finite game; although the results of this section are already discovered in [14], we use a different method. Sections 5 and 6 contain weak and strong continuity properties of the core correspondence. Section 8 gives some information on σ-continuous games. We also answer a question raised in [11]. Section 9 is devoted to selection theorems. For an application of selection theorems to mathematical economics we refer to a forthcoming paper of Sondermann [15]. Sections 7 and 10 give some approximation theorems; the purpose of these theorems is to approximate a game with infinitely many players by a game with finitely many players. Section 11 is an attempt to describe the structure of the core of σ-continuous games. We hope that our results will lead to a better understanding of the core of a game with infinitely many players. We know that for finite games our results are not general but the paper has to be seen as a paper on infinite games.

I thank Dr. D. Sondermann for giving me this problem and for the many discussions on the topic—in fact this paper is the solution of a problem raised while he was preparing [15].

1. Definitions and Notations

Definition 1. If Ω is a set and $\mathscr{A}$ is a family of subsets of Ω, then $\mathscr{A}$ is called an *algebra* if

(i) $\emptyset \in \mathscr{A}$;

(ii) $A \in \mathscr{A}$ implies $\Omega \backslash A = A^c \in \mathscr{A}$;

(iii) A_1 and $A_2 \in \mathscr{A}$ implies $A_1 \cup A_2 \in \mathscr{A}$.

$\mathscr{A}$ is called a *tribe* or σ-algebra, if $\mathscr{A}$ satisfies (i), (ii), and

(iv) if $(A_n)_{n \in \mathbb{N}} \in \mathscr{A}$, then $\bigcup_n A_n \in \mathscr{A}$.

A *measurable space* is a couple $(\Omega, \mathscr{A})$ such that $\mathscr{A}$ is a tribe on Ω.

Definition 2. A *game* is a triple $(\Omega, \mathscr{A}, v)$ where $\mathscr{A}$ is an algebra on Ω and v is a function

$$v\colon \mathscr{A} \to \mathbb{R}_+$$

such that $v(\emptyset) = 0$.

The points of Ω are sometimes called *players*, the elements of $\mathscr{A}$ are called *coalitions*.

The set of all games on $(\Omega, \mathscr{A})$ is denoted by $V(\Omega, \mathscr{A})$.

A *finite game* is a game $(\Omega, \mathscr{A}, v)$ where Ω is a finite set and $\mathscr{A}$ is the algebra consisting of all subsets of Ω.

Definition 3. If $A \in \mathscr{A}$, then the game $(A, \mathscr{A} \cap A, v_{/\mathscr{A} \cap A})$ is called the *restriction of v to the coalition A.*

If $\mathscr{B}$ is a subalgebra of $\mathscr{A}$, then we say that $(\Omega, \mathscr{B}, v_{/\mathscr{B}})$ is a *subgame* of $(\Omega, \mathscr{A}, v)$.

On $V(\Omega, \mathscr{A})$ we shall always use the following topology: a neighborhood base of $v_0 \in V(\Omega, \mathscr{A})$ is given by sets of the form

$$\{v \mid \forall j = 1,..., n \mid v_0(A_j) - v(A_j)| < \epsilon\},$$

where $A_j \in \mathscr{A}$ and $\epsilon > 0$.

The set of functions $f\colon \Omega \to \mathbb{R}$ such that f is the uniform limit of step-functions $\sum_{i=1}^{s} \lambda_i 1_{A_i}$ is denoted by $B(\Omega, \mathscr{A})$. Endowed with the norm

$$\|f\| = \sup_{\omega \in \Omega} |f(\omega)|$$

B is a Banach space.

The dual of B denoted by $\mathrm{ba}(\Omega, \mathscr{A})$ consists of all bounded additive measures on $(\Omega, \mathscr{A})$ ([3]).

DEFINITION 4. An *outcome* of a game $(\Omega, \mathscr{A}, v)$ is an element $\mu \in \mathrm{ba}(\Omega, \mathscr{A})$ such that

$$\forall A \in \mathscr{A}: \quad \mu(A) \geqslant v(A) \qquad \text{and} \qquad \mu(\Omega) = v(\Omega).$$

The set of all outcomes of a game $(\Omega, \mathscr{A}, v)$ is denoted by $\mathscr{C}(\Omega, \mathscr{A}, v)$ or $\mathscr{C}(v)$.

It follows immediately from the results of Ky Fan [4] that $\mathscr{C}(v)$ is non-empty if and only if

$$\text{for all } \lambda_1, ..., \lambda_n \geqslant 0 \text{ and all } A_1, ..., A_n \in \mathscr{A} \text{ such that}$$

$$\sum_{i=1}^{n} \lambda_i l_{A_i} \leqslant l_\Omega \qquad \text{we have} \qquad \sum \lambda_i v(A_i) \leqslant v(\Omega).$$

(Schmeidler [12] proved this theorem using other methods. The first to apply Ky Fan's results in game theory was Kannai [6].)

DEFINITION 5. (a) A game $(\Omega, \mathscr{A}, v)$ is called *balanced* if for all $\lambda_1, ..., \lambda_n \geqslant 0$ and all $A_1 \cdots A_n \in \mathscr{A}$ such that

$$\sum_{i=1}^{n} \lambda_i l_{A_i} \leqslant l_\Omega$$

we have that

$$\sum_{i=1}^{n} \lambda_i v(A_i) \leqslant v(\Omega).$$

The set of all balanced games is denoted by $V_b(\Omega, \mathscr{A})$ or V_b.

(b) A game is *totally balanced* if for all $\lambda_1, ..., \lambda_n \geqslant 0$, $A_0, A_1, ..., A_k \in \mathscr{A}$ such that

$$\sum_{i=1}^{n} \lambda_i l_{A_i} \leqslant l_{A_0}$$

we have

$$\sum_{i=1}^{n} \lambda_i v(A_i) \leqslant v(A_0).$$

The set of all totally balanced games is denoted by V_{tb}.

(c) A game is called *exact* if $\forall A \in \mathscr{A}$

$$v(A) = \min\{\mu(A) \mid \mu \in \mathscr{C}(v)\}.$$

(The minimum really exists since $\mathscr{C}(v)$ is $\sigma(\mathrm{ba}, B)$-compact). The set of all exact games is denoted by V_e.

(d) A game is called *convex* if A_1 and $A_2 \in \mathscr{A}$ imply

$$v(A_1) + v(A_2) \leqslant v(A_1 \cup A_2) + v(A_1 \cap A_2).$$

The set of all convex games is denoted by V_c.

It is clear that V_b, V_{tb} and V_c are closed in V. The proof that also V_e is closed will be given later.

It is well known (see, e.g., Schmeidler [11]) that

$$V_c \subset V_e \subset V_{tb} \subset V_b .$$

For the sake of completeness and to illustrate the method we shall use further on, we prove that $V_c \subset V_{tb}$.

LEMMA 1 [4]. *A convex game is totally balanced.*

Proof. Let $\lambda_1, \ldots, \lambda_n \geqslant 0$, $A_0, A_1, \ldots, A_n \in \mathscr{A}$ and

$$\sum_{i=1}^{n} \lambda_i l_{A_i} \leqslant l_{A_0} .$$

We can add some other terms of the form $\lambda 1_A$ and so obtain

$$\sum_{i=1}^{n} \lambda_i l_{A_i} = l_{A_0} .$$

It is clear that Lemma 1 will be proved if

$$\sum_{i=1}^{n} \lambda_i v(A_i) \leqslant v(A_0).$$

Let $\mathscr{B}$ be the set of all subsets of A_0 which are elements of the algebra generated by $A_1, \ldots, A_n$. On $\mathbb{R}^{\mathscr{B}}$ we define the linear function

$$\mathbb{R}^{\mathscr{B}} \xrightarrow{u} \mathbb{R}$$

$$(\lambda_A)_A \mapsto \sum_{A \in \mathscr{B}} \lambda_A v(A).$$

If K denotes the set $\{(\lambda_A) \in \mathbb{R}_+^{\mathscr{B}} \mid \sum \lambda_A \cdot l_A = l_{A_0}\}$ then using the fact that K is compact convex it is easily seen that $L = \{(\lambda_A) \in K \mid u(\lambda) = \max u(K)\}$ is non-empty and compact convex. To prove the lemma it is sufficient to prove that

$$\max_{\lambda\in L} \lambda_{A_0} = 1.$$

Suppose that

$$\lambda^0_{A_0} = \max_{\lambda\in L} \lambda_{A_0} < 1,$$

then there exists a covering $(A_i)_{i\in I}$ of A_0 such that $\min_{i\in I} \lambda^0_{A_i} = \gamma > 0$. (Otherwise $\sum_{A\in\mathscr{B}} \lambda_A{}^0 \cdot l_A \leqslant l_{A_0}$.) Now

$$\sum_{A\in\mathscr{B}} \lambda_A{}^0 v(A) = \lambda^0_{A_0} v(A_0) + \sum_{i\in I} \lambda_{A_i} v(A_i) + \sum_{\mathscr{B}\setminus\{A_i \mid i\in I\}} \lambda_A v(A).$$

The second term

$$\sum_{i\in I} \lambda^0_{A_i} v(A_i) = \sum_{i\in I} (\lambda^0_{A_i} - \gamma)\, v(A_i) + \gamma \sum_{i\in I} v(A_i)$$

and by convexity $\sum_{i\in I} v(A_i) \leqslant v(A_0) + \sum$ mixed terms. It is now clear that this proves

$$\max_{\lambda\in L} \lambda_{A_0} \geqslant \lambda^0_{A_0} + \gamma > \lambda^0_{A_0},$$

a contradiction to the choice of λ^0. So clearly $\max \lambda_{A_0} = 1$ and hence u reaches its maximum on l_{A_0}. This proves the lemma.

2. A Useful Characterization of Convex Games

Let f be a positive function of $B_+(\Omega, \mathscr{A})$ which can be written under the form $\sum_{i=1}^n \lambda_i l_{A_i}$. It is clear that there is one and only one way to write f in the form $\sum_{j=1}^s \mu_j l_{C_j}$ where $\mu_j > 0$ and $C_1 \supsetneqq C_2 \supsetneqq \cdots \supsetneqq C_s$. The expression $\sum_{j=1}^s \mu_j l_{C_j}$ is called the *canonical form* of f.

LEMMA 2. *If $(\Omega, \mathscr{A}, v)$ is a game, then the following conditions are all equivalent:*

(1) *v is convex.*

(2) *If $\alpha_1 \cdots \alpha_m \geqslant 0$, $B_1 \cdots B_m \in \mathscr{A}$, $\lambda_1 \cdots \lambda_n > 0$, $A_1 \cdots A_n \in \mathscr{A}$,*

$$\sum_{k=1}^m \alpha_k l_{B_k} \leqslant \sum_{i=1}^n \lambda_i l_{A_i} = f,$$

then

$$\sum_{k=1}^{m} \alpha_k v(B_k) \leqslant \sum_{j=1}^{s} \mu_j v(C_j),$$

where $\sum_{j=1}^{s} \mu_j l_{C_j}$ *is the canonical form of* f.

(3) *If* $f = \sum_{i=1}^{n} \lambda_i l_{A_i}$ *where* $\lambda_i \geqslant 0$, $A_i \in \mathscr{A}$, *then*

$$\min_{\mu \in \mathscr{C}(v)} \langle f, \mu \rangle = \sum_{j=1}^{s} \mu_j v(C_j)$$

where $\sum_{j=1}^{s} \mu_j l_{C_j}$ *is the canonical form of* f.

(4) *The same as in condition* (3) *but with* $s = 2$.

(5) *If* $C_1 \supsetneqq C_2 \supsetneqq \cdots \supsetneqq C_s$ *where* $C_j \in \mathscr{A}$, *then* $\exists \mu \in \mathscr{C}(v)$ *such that* $\mu(C_j) = v(C_j)$, $j = 1,\ldots, s$.

(6) *Same as* (5) *but with* $s = 2$.

Proof. The proof goes as follows;

$$\begin{array}{ccccccc} (1) & \Rightarrow & (2) & \Rightarrow & (3) & \Rightarrow & (4) \\ & & & & \Downarrow & & \Downarrow \\ & & & & (5) & \Rightarrow & (6) \Rightarrow (1). \end{array}$$

(1) $\Rightarrow$ (2). As in Lemma 1 we may suppose that

$$\sum_{k=1}^{m} \alpha_k l_{B_k} = f.$$

The proof is done by induction on s. For $s = 1$ this is nothing other than Lemma 1. Suppose the theorem to be true for $s - 1$.

As in the proof of Lemma 1 we introduce $\mathscr{B}$, the set of subsets of $C_A = \{\omega \mid f(\omega) > 0\}$ which are elements of the algebra generated by $A_1 \cdots A_n$, $B_1 \cdots B_k$. Let

$$K = \left\{(\lambda_A)_{A\in\mathscr{B}} \mid (\lambda_A) \in \mathbb{R}_+{}^{\mathscr{B}} \sum_{A\in\mathscr{B}} \lambda_A l_A = f\right\},$$

$$L = \left\{(\lambda_A) \in K \mid \sum \lambda_A v(A) = \max u(K)\right\},$$

where u is the linear form

$$u\colon \mathbb{R}^{\mathscr{B}} \to \mathbb{R}; \qquad (\lambda_A) \to \sum_{\mathscr{B}} \lambda_A v(A).$$

As in the proof of Lemma 1 we can easily show that

$$\max_{\lambda \in L} \lambda_{C_A} = \min_{\Omega} f.$$

So let $\lambda^0 \in L$ be such that $\lambda^0_{C_1} = \min f = \mu_1$. Clearly

$$\sum_{k=1}^{m} \alpha_k v(B_k) \leqslant \lambda^0_{C_1} v(C_1) + \sum_{\substack{A \in \mathscr{B} \\ A \neq C_1}} \lambda_A{}^0 v(A).$$

But the canonical form of $\sum_{A \in \mathscr{B}, A \neq C_1} \lambda_A{}^0 l_A$ is nothing other than $\sum_{j=2}^{s} \mu_j l_{C_j}$ and hence by the induction step

$$\sum_{\substack{A \in \mathscr{B} \\ A \neq C_1}} \lambda_A{}^0 v(A) \leqslant \sum_{j=2}^{s} \mu_j v(C_j).$$

So finally

$$\sum_{k=1}^{m} a_k v(B_k) \leqslant \sum_{j=1}^{s} \mu_j v(C_j).$$

(2) $\Rightarrow$ (3). By Ky Fan's Theorem 12, Corollary 6 [4], we have that

$$\min\langle f, \mu\rangle = \sup \left\{ \sum_{i=1}^{m} \alpha_i v(B_i) - v(\Omega) \left\| f - \sum_{i=1}^{m} \alpha_i l_{B_i} \right\| ; \alpha_i > 0, B_i \in \mathscr{A} \right\}.$$

Suppose that

$$\left\| f - \sum_{i=1}^{m} \alpha_i l_{B_i} \right\| = \lambda_0,$$

then clearly

$$\sum_{i=1}^{m} \alpha_i l_{B_i} \leqslant \lambda_0 l_\Omega + \sum_{j=1}^{s} \mu_j l_{C_j}.$$

Hence by (2) we have

$$\sum_{i=1}^{m} \alpha_i v(B_i) \leqslant \lambda_0 v(\Omega) + \sum_{j=1}^{s} \mu_j v(C_j).$$

So finally

$$\sup\{\cdots\} = \sum_{j=1}^{s} \mu_j v(C_j).$$

(3) $\Rightarrow$ (4) trivial.

(3) $\Rightarrow$ (5) as well as (4) $\Rightarrow$ (6). Let $f = \sum_{j=1}^{s} l_{C_j}$ and let ν_0 be such that

$$\nu_0 \in \mathscr{C}(x) \qquad \text{and} \qquad \langle \nu_0, f\rangle = \min_{\nu \in \mathscr{C}(v)} \langle f, \nu\rangle.$$

Then

$$\sum_{j=1}^{s} v(C_j) = \sum_{j=1}^{s} \nu_0(C_j)$$

but $\nu_0(C_j) \geqslant v(C_j)$, $\forall j = 1,..., s$, implies $v(C_j) = \nu_0(C_j)$, $\forall_j = 1,..., s$.

(6) $\Rightarrow$ (1). Let A_1 and $A_2 \in \mathscr{A}$ and let $C_1 = A_1 \cup A_2$, $C_2 = A_1 \cap A_2$. Let μ be such that $\mu(A_1 \cup A_2) = v(A_1 \cup A_2)$ and $\mu(A_1 \cap A_2) = v(A_1 \cap A_2)$. Clearly

$$\begin{aligned} v(A_1) + v(A_2) &\leqslant \mu(A_1) + \mu(A_2) \\ &= \mu(A_1 \cup A_2) + \mu(A_1 \cap A_2) \\ &\leqslant v(A_1 \cup A_2) + v(A_1 \cap A_2). \end{aligned}$$

COROLLARY 1. *$\forall f \in B(\Omega, \mathscr{B})$ where $\mathscr{B}$ is a subalgebra of $\mathscr{A}$, the functions* V: $V_c \to \mathbb{R}$,

$$v \mapsto v(f) = \min\{\langle f, \mu\rangle \mid \mu \in \mathscr{C}(v)\},$$

$\bar{V}$: $V_c \to \mathbb{R}$,

$$v \mapsto \bar{v}(f) = \max\{\langle f, \mu\rangle \mid \mu \in \mathscr{C}(v)\},$$

are continuous and depend only on the restriction of v to $\mathscr{B}$.

Proof. The corollary is proved for V and for functions $f = \sum_{i=1}^{m} \alpha_i l_{B_i}$ where $\alpha_i > 0$ by adding a constant γ to an arbitrary function and subtracting $\gamma\, v(\Omega)$; afterwards, we prove it for arbitrary step functions. The density of the stepfunctions in $B(\Omega, \mathscr{B})$ then gives the result.

COROLLARY 2. *If $f \in B(\Omega, \mathscr{B})$ where $\mathscr{B}$ is a subalgebra of $\mathscr{A}$ and $(\Omega, \mathscr{A}, v)$ is a convex game, then*

$$\min\langle f, \mathscr{C}(\Omega, \mathscr{B}, v)\rangle = \min\langle f, \mathscr{C}(\Omega, \mathscr{A}, v)\rangle.$$

COROLLARY 3. *Let $(\Omega, \mathscr{A}, v)$ be a convex game. If $(C_i)_{i\in I}$ is a totally ordered family of elements of $\mathscr{A}$, then $\exists \mu \in \mathscr{C}(v)$ such that*

$$\mu(C_i) = v(C_i), \qquad \forall i \in I.$$

Proof. $\forall J \subset I$, J finite, we put

$$K_J = \{\mu \mid \mu \in \mathscr{C}(v)\; \mu(C_i) = v(C_i), \forall i \in J\}.$$

By point 3 of the lemma K_J is non-empty and clearly is a compact subset of $\mathscr{C}(v)$. Since $K_J \supset K_{J'}$ if $J' \supset J$ we also have that $(K_J)_{J\subset I, J\text{ finite}}$ satisfies the finite intersection property, hence $\bigcap_J K_J$ is non-empty [7]. Clearly every element of this intersection satisfies $\mu(C_i) = v(C_i)$, $\forall i \in I$.

3. The Projection Theorem

If $(\Omega, \mathscr{B}, v)$ is a subgame of $(\Omega, \mathscr{A}, v)$, then every element $\mu \in \mathscr{C}(\Omega, \mathscr{A}, v)$ defines (by restriction to $\mathscr{B}$) an element of $\mathscr{C}(\Omega, \mathscr{B}, v)$. It is easy to give examples where the set of restrictions $K(\mathscr{B})$ is actually different from $\mathscr{C}(\Omega, \mathscr{B}, v)$. The reason for this is explained by the following propositions:

Proposition 1. *If $(\Omega, \mathscr{A}, v)$ is a balanced game then the next statements are equivalent:*

(i) *v is exact.*

(ii) *$K(\mathscr{B}) = \mathscr{C}(\Omega, \mathscr{B}, v)$ for each algebra $\mathscr{B}$ generated by one element of $\mathscr{A}$.*

Proof. (i) $\Rightarrow$ (ii). Suppose $\mathscr{B} = \{\emptyset, A, A^c, \Omega\}$, then since v is balanced $\mathscr{C}(\Omega, \mathscr{B}, v)$ is the segment joining the points $(v(A), v(\Omega) - v(A))$ and $(v(\Omega) - v(A^c), v(A^c))$. Since $K(\mathscr{B})$ is convex and contains by exactness these two points, we have $K(\mathscr{B}) = \mathscr{C}(\Omega, \mathscr{B}, v)$.

(ii) $\Rightarrow$ (i). Let $A \in \mathscr{A}$ and let $\mathscr{B} = \{\emptyset, A, A^c, \Omega\}$. In $\mathscr{C}(\Omega, \mathscr{B}, v)$ there is an element μ such that $\mu(A) = v(A)$. Since $K(\mathscr{B}) = \mathscr{C}(\Omega, \mathscr{B}, v)$, μ is the projection of $\bar{\mu}$. Clearly $\bar{\mu}(A) = v(A)$.

Proposition 2. *If $(\Omega, \mathscr{A}, v)$ is balanced then the next statements are equivalent:*

(i) *v is convex.*

(ii) *$\forall \mathscr{B}$ subalgebra of $\mathscr{A}$ $K(\mathscr{B}) = \mathscr{C}(\Omega, \mathscr{B}, v)$.*

(iii) *$\forall \mathscr{B}$ generated by at most 2 elements of $\mathscr{A}$: $K(\mathscr{B}) = \mathscr{C}(\Omega, \mathscr{B}, v)$.*

Proof. (i) $\Rightarrow$ (ii). Suppose that $\mu \in \mathscr{C}(\Omega, \mathscr{B}, v) \backslash K(\mathscr{B})$, then by the Hahn–Banach separation theorem $\exists f \in B(\Omega, \mathscr{B})$ such that

$$\langle f, \mu \rangle < \min\{\langle f, \nu \rangle \mid \nu \in K(\mathscr{B})\},$$

since

$$\begin{aligned}\min\{\langle f, \nu \rangle \mid \nu \in K(\mathscr{B})\} &= \min\{\langle f, \nu \rangle \mid \nu \in \mathscr{C}(\Omega, \mathscr{A}, v)\} \\ &= \min\{\langle f, \nu \rangle \mid \nu \in \mathscr{C}(\Omega, \mathscr{B}, v)\},\end{aligned}$$

we obtain a contradiction.

(ii) $\Rightarrow$ (iii) trivial.

(iii) $\Rightarrow$ (i). By the preceding proposition the game $(\Omega, \mathscr{A}, v)$ is already exact. It follows that $\forall \mathscr{B}$ subalgebra $(\Omega, \mathscr{B}, v)$ is also exact. Suppose now that $\mathscr{B}$ is generated by A_1 and A_2, then $\mathscr{B}$ is also generated by the partition $A_1 \cap A_2$, $A_1 \triangle A_2$, $(A_1 \cup A_2)^c$. Since $(\Omega, \mathscr{B}, v)$ is exact and has 3 players, it is convex. Indeed let B_1 and $B_2 \in \mathscr{B}$.

(1) $B_1 \cap B_2$ is non-empty and $B_1 \cup B_2 = \Omega$, then let $\mu \in \mathscr{C}(\Omega, \mathscr{B}, v)$ be such that $\mu(B_1 \cap B_2) = v(B_1 \cap B_2)$. Clearly

$$v(B_1) + v(B_2) \leqslant \mu(B_1) + \mu(B_2) = \mu(\Omega) + \mu(B_1 \cap B_2) = v(\Omega) + v(B_1 \cap B_2).$$

(2) The other cases may be checked in the same way. Since $(\Omega, \mathscr{B}, v)$ is convex, $\exists \mu \in \mathscr{C}(\Omega, \mathscr{B}, v)$ such that

$$\mu(A_1 \cup A_2) = v(A_1 \cup A_2) \quad \text{and} \quad \mu(A_1 \cap A_2) = v(A_1 \cap A_2).$$

By (iii) μ is the projection of an element of $\mathscr{C}(\Omega, \mathscr{A}, v)$ and hence 6 of Lemma 2 is verified.

COROLLARY. *If $(\Omega, \mathscr{A}, v)$ is a balanced game such that $K(\mathscr{B}) = \mathscr{C}(\Omega, \mathscr{B}, v)$ for all subalgebra generated by a partition of Ω in exactly 3 elements of $\mathscr{A}$, then the exact envelop of v is convex.* (See [11] for the definition of exact envelop.)

Proof. This follows immediately from the proof (iii) $\Rightarrow$ (i).

4. Extreme Points of Cores of Finite Convex Games

Let $(\Omega, \mathscr{A}, v)$ be a finite convex game with n players. The *core* $\mathscr{C}(v)$ is then a subset of $\mathbb{R}^\Omega$ and is the intersection of a finite number of half spaces $\{x \in \mathbb{R}^\Omega \mid \sum_{i \in S} x_i \geqslant v(S)\}$, $S \subset \Omega$, and $\{x \mid \sum_{\omega \in \Omega} x_\omega \leqslant v(\Omega)\}$. It follows that the set ex $\mathscr{C}(v)$ of extreme points of $\mathscr{C}(v)$ is discrete and that $\mathscr{C}(v)$ is the convex hull of ex $\mathscr{C}(v)$. To find such an extreme point we proceed as follows. We first order Ω, i.e., $\Omega = \{\omega_1, ..., \omega_n\}$. Then we introduce the finite chain

$$C_1 = \Omega, \qquad C_2 = \{\omega_2, ..., \omega_n\}, \qquad C_i = \{\omega_j \mid j \geqslant i\}, \qquad C_n = \{\omega_n\}.$$

By 3 of Lemma 2 $\exists \mu \in \mathscr{C}(v)$ such that $\mu(C_j) = v(C_j)$. Since C_j, $j \leqslant n$, is a generating set of $\mathscr{A}$ it follows that the numbers $v(C_j)$ completely determine μ. It is very easily seen that μ is extreme. Indeed, suppose that $\mu = \frac{1}{2}(\mu_1 + \mu_2)$, then $\forall j$ we have

$$\mu(C_j) = \tfrac{1}{2}(\mu_1(C_j) + \mu_2(C_j)) = v(C_j)$$

since μ_1 and μ_2 belong to the core $\mathscr{C}(v)$: $\mu_1(C_j) = \mu_2(C_j) = v(C_j)$, hence $\mu = \mu_1 = \mu_2$. Conversely, every extreme point of $\mathscr{C}(v)$ can be obtained in such a way. To see this we introduce the polyhedron

$$K = \left\{ x \in \mathbb{R}^\Omega \mid \forall S \subset \Omega \sum_{\omega \in S} x_\omega \geqslant v(S), \sum_{\omega \in \Omega} x_\omega \leqslant 2v(\Omega) \right\}.$$

It can be checked that an extreme point x of K for which $\sum_{\omega \in \Omega} x_\omega = v(\Omega)$ is also an extreme point of $\mathscr{C}(v)$ and conversely. So let $x \in \operatorname{ex} \mathscr{C}(v)$. By the Minkowski separation theorem $\exists f \in \mathbb{R}^\Omega$ such that

$$f(x) < f(y), \qquad \forall y \in K, \qquad y \neq x.$$

We now write f in its canonical form $f = \sum_{j=1}^{s} \mu_j C_j$. By the strict inequality and the definition of K we obtain immediately $C_1 = \Omega$. However, we cannot prove that $s = n$. To obtain a separating function f such that $s = n$ we apply the discreteness of the set of extreme points. So we can modify f in such a way that in its canonical form $s = n$. It then follows that $\forall j$ $v(C_j) = \sum_{\omega \in C_j} x_\omega$. Hence x can be obtained by the complete ordering $>$ defined as $\omega' > \omega$ if and only if $\exists j$ with $\omega' \in C_j$ and $\omega \notin C_j$.

Remark 1. The fact that we cannot prove $s = n$ is due to some kind of additivity of v. Suppose $s < n$, then $\exists i$ such that $C_i \backslash C_{i+1}$ contains at least 2 elements. So we can find $A_1 \neq A_2$ such that $C_i \supsetneqq A_1 \supsetneqq C_{i+1}$, $C_i \supsetneqq A_2 \supsetneqq C_{i+1}$ as well as $A_1 \cap A_2 = C_{i+1}$, $A_1 \cup A_2 = C_i$. It then follows that $\exists \mu_1$ $\mu_1(A_1) = v(A_1)$, $\mu_1(C_k) = v(C_k)$ $\forall k$ and $\exists \mu_2$ $\mu_2(A_2) = v(A_2)$, $\mu_2(C_k) = v(C_k)$ $\forall k$. Since $\mu(C_k) = v(C_k)$ defines an extreme point, we have $\mu = \mu_1 = \mu_2$ and hence

$$v(C_i) - v(A_1) = \mu_1(C_i \backslash A_1) = \mu_2(C_i \backslash A_1) = \mu_2(A_2 \backslash C_{i+1}) = v(A_2) - v(C_{i+1}),$$

so clearly

$$v(C_i) + v(C_{i+1}) = v(A_1) + v(A_2).$$

Remark 2. See also [14] for a discussion of this topic.

5. Weak Continuity Properties of the Core

Before giving the results let us recall some definitions. A correspondence ϕ from a topologic space X into a topologic space Y is called *upper semicontinuous* if $\forall 0$ open in Y $\{x \mid \phi(x) \subset 0\}$ is open in X. It is called *lower semicontinuous* if $\forall 0$ open in Y $\{x \mid \phi(x) \cap 0 \neq \emptyset\}$ is open in X.

PROPOSITION 3. *The correspondence* $\mathscr{C}: V(\Omega, \mathscr{A}) \to \mathrm{ba}(\Omega, \mathscr{A})$ *is upper semicontinuous for the weak topology* $\sigma(\mathrm{ba}, B)$ *on* ba.

Proof. Since every point $v_0 \in V$ has a neighborhood on which $\mathscr{C}$ remains in a $\sigma(\mathrm{ba}, B)$ compact set, we only have to prove the graph of $\mathscr{C}$ to be closed. So let $\mu_\alpha \in \mathscr{C}(v_\alpha)$, $v_\alpha \to v$, $\mu_\alpha \to \mu$. Since $\mu_\alpha(\Omega) = v_\alpha(\Omega) \to v(\Omega)$, we have $\mu(\Omega) = v(\Omega)$. But $\mu_\alpha(A) \geqslant v_\alpha(A)$, $\forall \alpha$, $\forall A \in \mathscr{A}$, and hence $\mu(A) \geqslant v(A)$, $\forall A$. This proves $\mu \in \mathscr{C}(v)$.

COROLLARY 1. V_e *is a closed subset of* V.

Proof. Let $v_\alpha \in V_e$ and $v_\alpha \to v$. Let $A \in \mathscr{A}$. $\forall \alpha\, \exists \mu_\alpha \in \mathscr{C}(v_\alpha)$ with $\mu_\alpha(A) = v_\alpha(A)$. Let μ be an adherent point to μ_α, then $\mu(A) = v(A)$ and $\mu \in \mathscr{C}(v)$.

PROPOSITION 4. *The correspondence* $\mathscr{C}: V_c \to \mathrm{ba}(\Omega, \mathscr{A})$ *is continuous.*

Proof. Upper semicontinuity is already proved in the preceding proposition. Let 0 be an open set in $\mathrm{ba}(\Omega, \mathscr{A})$, $\sigma(\mathrm{ba}, B)$. Let v_0 be such that $\mathscr{C}(v_0) \cap 0 \neq \emptyset$. We have to prove that if $v_\alpha \to v$ then $\exists \alpha_0$, $\forall \alpha \geqslant \alpha_0$, $\mathscr{C}(v_\alpha) \cap 0 \neq \emptyset$.

Let $\mu_0 \in \mathscr{C}(v_0) \cap 0$, choose $\epsilon > 0$, and choose the partition $A_1 \cdots A_n$ of Ω such that

$$0' = \left\{ \mu \,\Big|\, \sum_{j=1}^{n} | \mu_0(A_j) - \mu(A_j)| < \epsilon \right\} \subset 0.$$

Suppose now that $v_\alpha \to v_0$ and $\mathscr{C}(v_\alpha) \cap 0' = \emptyset\ \forall \alpha$. By the projection theorem it then follows that, if $\mathscr{B}$ is the algebra generated by $A_1 \cdots A_n$,

$$\mathscr{C}(\Omega, \mathscr{B}, v) \cap 0'' = \emptyset$$

(where $0''$ has the same meaning as $0'$ but with $\mu \in \mathrm{ba}(\Omega, \mathscr{B})$ and μ_0 replaced by $\mu_0/\mathscr{B}$). By the separation theorem $\exists \lambda_1^\alpha \cdots \lambda_n^\alpha$ $\max | \lambda_j | = 1$ such that

$$\sum_{j=1}^{n} \lambda_j^\alpha \mu_0(A) + \epsilon < \min_{\mu \in \mathscr{C}(v_\alpha)} \left\langle \sum_{j=1}^{n} \lambda_j^\alpha I_{A_j}, \mu \right\rangle$$

λ^α has a cluster point λ in $\mathbb{R}^n$ such that $\max | \lambda_j | = 1$. Hence by continuity of the left-hand side

$$\sum_{j=1}^{n} \lambda_j \mu_0(A) + \epsilon \leqslant \min_{\mu \in \mathscr{C}(v)} \left\langle \sum_{j=1}^{n} \lambda_j I_{A_j}, \mu \right\rangle.$$

Since $\epsilon > 0$ this is a contradiction.

A COUNTEREXAMPLE. *The correspondence* $\mathscr{C}$: $V_{tb} \to \mathrm{ba}(\Omega, \mathscr{A})$ *is* (*in general*) *not lower semicontinuous.*

Let $\Omega = \mathbb{N}$, $\mathscr{A} = \{S \mid S \subset \mathbb{N}\}$.
Let v^n be a game defined on $\{1,..., n\}$.

$$\begin{aligned} v^n(A) &= 0 && \text{if} \quad \# A^c \geqslant 2 \\ &= \frac{n-1}{n} && \text{if} \quad \# A^c = 1 \\ &= 1 && \text{if} \quad \# A^c = 0. \end{aligned}$$

Let $v_n \in V_{tb}(\mathbb{N}, 2^{\mathbb{N}})$ be defined as

$$v_n(S) = v^n(S \cap \{1,..., n\}).$$

It is easily seen that

$$\begin{aligned} v_n(S) &\to v(S) = 0 && \text{if} \quad \# S^c \geqslant 2, \\ v_n(S) &\to v(S) = 1 && \text{if} \quad \# S^c = 1, \\ v_n(\mathbb{N}) &\to v(\mathbb{N}) = 1. \end{aligned}$$

Let $x \in \beta\mathbb{N} \setminus \mathbb{N}$ ($\beta\mathbb{N}$ is the Stone Čech compactification of $\mathbb{N}$) [7] and "choose" x such that

$$x \in \overline{2\mathbb{N}} \setminus \overline{2\mathbb{N} + 1}.$$

Clearly $\delta_x \in \mathscr{C}(v)$ but δ_x is not adherent to $\mathscr{C}(v_n)$. Indeed $\mathscr{C}(v_n)$ has only one member $\mu_n(s) = 1/n$ if $s \leqslant n$, $\mu_n(0) = 0$ if $s > n$. Hence, if $S = 2\mathbb{N}$, $\mu_n(S) \to \frac{1}{2}$ and so μ is not adherent to μ_n.

6. STRONG CONTINUITY PROPERTIES OF THE CORE

If $V_c(\Omega, \mathscr{A})$ is the set of all convex games defined on $(\Omega, \mathscr{A})$, then we can put some kind of strong topology on V_c. This topology will be induced by the metric

$$d(v, w) = \sup_{A \in \mathscr{A}} | v(A) - w(A)| .$$

The following theorem says that, also with respect to this strong topology and the strong (= norm) topology on ba, the core behaves nicely.

PROPOSITION 5. $\mathscr{C}: (V_c, d) \to (\mathrm{ba}(\Omega, \mathscr{A}), \| \ \|)$ *is a convex closed continuous correspondence.*

Proof. (a) *Upper semicontinuity.* Let $\epsilon > 0$, $v \in V_c$, $v_n \to v$; we have to prove that $\exists n_0$ such that, $\forall n \geqslant n_0$, $\mathscr{C}(v_n) \subset \mathscr{C}(v) + S(0, \epsilon)$ (where $S(0, \epsilon)$ denotes the closed ϵ ball around 0 in ba). Suppose on the contrary that there is a subsequence (still denoted by v_n) such that $\forall n\ \exists \mu_n \in \mathscr{C}(v_n) \backslash [\mathscr{C}(v) + S(0, \epsilon)]$. By the Hahn–Banach separation theorem and the $\sigma(\text{ba}, B)$ compactness of $\mathscr{C}(v) + S(0, \epsilon)$ there is a step function f_n such that $\| f_n \| = 1$ and

$$\langle f_n, \mu_n \rangle < \min \langle f_n, \mathscr{C}(v) + S(0, \epsilon) \rangle,$$

hence

$$\langle f_n, \mu_n \rangle < \min \langle f_n, \mathscr{C}(v) \rangle - \epsilon.$$

Let $\sum_{j=1}^{s} a_j^n l_{C_j^n}$ be the canonical form of $f_n + 1$. Now

$$\begin{aligned} \min \langle f_n, \mathscr{C}(v) \rangle &= \min \langle f_n + 1, \mathscr{C}(v) \rangle - v(\Omega) \\ &= \sum_{j=1}^{s} a_j^n v(C_j^n) - v(\Omega) \end{aligned}$$

for n large enough

$$\begin{aligned} \langle f_n, \mathscr{C}(v) \rangle &\leqslant \sum_{j=1}^{s} a_j^n v_n(C_j^n) + \frac{\epsilon}{4} \sum_{j=1}^{s} a_j^n - v_n(\Omega) + \frac{\epsilon}{4} \\ &\leqslant \min \langle f_n, \mathscr{C}(v_n) \rangle + \frac{\epsilon}{2}. \end{aligned}$$

Hence for n large enough

$$\langle f_n, \mu_n \rangle < \min \langle f_n, \mathscr{C}(v_n) \rangle - \epsilon/2,$$

a contradiction.

(b) *Lower semicontinuity* is proved in the same way: Indeed we have to prove that, for all $v_n \to v$, $\forall \mu_0 \in \mathscr{C}(v)$, $\forall \epsilon > 0$, $\exists n_0$ such that $\forall n \geqslant n_0$, $\exists \mu_n \in \mathscr{C}(v_n) \cap S(\mu_0, \epsilon)$. Suppose this is not possible. By passing to a subsequence we may suppose that $\forall n\ \mathscr{C}(v_n) \cap S(\mu_0, \epsilon) = \emptyset$. By the separation theorem $\exists f_n \| f_n \| = 1 \| f_n \|$, a step function such that

$$\min \langle f^n, \mathscr{C}(v_n) \rangle > \sup \langle f^n, S(\mu_0, \epsilon) \rangle.$$

For n large enough this implies in the same way as above

$$\min \langle f^n, \mathscr{C}(v_n) \rangle > \langle f^n, \mu_0 \rangle + \epsilon/2,$$

a contradiction.

PROPOSITION 6. *There is a strong* ($=$ norm) *continuous function* $f: V_c \to \text{ba}(\Omega, \mathscr{A})$ *such that* $\forall v \in V_c$, $f(v) \in \mathscr{C}(v)$.

Proof. Follows from Michael's selection theorem [9].

COROLLARY. *If $Y \subset V_c$ is closed for the distance d introduced above and if $f: Y \to \mathrm{ba}$ is a* norm *continuous selection of $\mathscr{C}$: $Y \to \mathrm{ba}$, then $\exists g$: $V_c \to \mathrm{ba}$, a* norm *continuous selection of $\mathscr{C}$ such that $g/Y = f$.*

Proof. This follows from the results of [9].

A COUNTEREXAMPLE. Let

$$\Omega = [01], \qquad \mathscr{A} = \{A \mid A \subset [01]; A \text{ Borel}\}.$$

$$f_0: [01] \to [01]\; x \to \frac{e^x - 1}{e - 1} \quad \text{(see Section 12).}$$

$f_n[01] \to [01]$ is defined as

$$f_n(0) = 0, \qquad f_n\left(\frac{1}{2n}\right) = \frac{1}{2n}.$$

$$f_n(x) = f_0(x), \qquad \frac{1}{n} \leqslant x \leqslant 1.$$

On $[0, 1/2n]$ and $[1/2n, 1/n]$ f_n is linear. v_n is now defined as

$$v_n(A) = f_n(\lambda(A)) \text{ with } \lambda \text{ Lebesgue measure,}$$
$$v_0(A) = f_0(\lambda(A)).$$

Clearly $v_n \to v_0$ strongly, but $\mathscr{C}(v_n) = \{\lambda\}$. Indeed, if $\mu \in \mathscr{C}(v_n)$, then $\forall S\; \mu(S) = \sum_{k=1}^{s} \mu(S_k)$, where S_k is a partition of S into elements of measure smaller than $1/2n$. So $\mu(S) = \sum \mu(S_k) \geqslant \sum \lambda(S_k) = \lambda(S)$ (remark that $\mu \in \mathscr{C}(v_n)$). Hence $\mu = \lambda$.

In Section 12 it is proved that $\mathscr{C}(v_0)$ has other elements than λ. From this counterexample it follows that in general $\mathscr{C}$: $V_b \to \mathrm{ba}$ is not lower semi-continuous.

ANOTHER COUNTEREXAMPLE. Let $(\Omega, \mathscr{A}, v_0)$ be as in the last counter-example. Let $f_n[01] \to \mathbb{R}_+$ be defined as

$$f_n(x) = f_0(x), \qquad \text{if} \quad 0 \leqslant x < 1 - \frac{1}{n}.$$

$$f_n(x) = x, \qquad \text{if} \quad 1 - \frac{1}{n} \leqslant x \leqslant 1,$$

$$v_n(S) = f_n(\lambda(S)).$$

We prove $\mathscr{C}(v_n) = \{\lambda\}$ where λ is Lebesgue measure. Observe that, if $\lambda(S) < 1/n$, then $\mu(S) \leqslant \lambda(S)$ for all $\mu \in \mathscr{C}(v_n)$ (otherwise S^c can block hence

$\forall S\, \lambda(S) \geqslant \mu(S)$) and this implies $\mu = \lambda$. It is clear that $v_n \to v_0$ strongly since $f_n \to f_0$ uniformly. However one can easily see that

$$\sup \left\{ \sum \lambda_i \mid v_n(S_i) - v_0(S_i) \mid \lambda_i l_{S_i} \leqslant 1, \lambda_i \geqslant 0 \right\}$$

$$= \frac{n}{n-1} \left[1 - \frac{1}{n} - \frac{e^{1-(1/n)} - 1}{e - 1} \right] \to 0 \qquad \text{as} \qquad n \to \infty,$$

and this implies that $\mathscr{C}$ is not lower semicontinuous even for stronger topologies on V_b.

7. Approximation

Let I be a directed set and let $(\mathscr{A}_i)_{i \in I}$ be a corresponding increasing family of subalgebras of $\mathscr{A}$. For all $i \in I$ we define the upper semicontinuous correspondence

$$K_i\colon V_c \to ba_+(\Omega, \mathscr{A})$$

$$v \to \{\mu \mid \mu(\Omega) = v(\Omega) \text{ and } \forall A \in \mathscr{A}_i,\, v(A) \leqslant \mu(A)\}.$$

Proposition 7. *If $\mathscr{A} = \bigcup_I \mathscr{A}_i$ then $K_i \downarrow \mathscr{C}$ uniformly on compact sets of V_c (for the weak topology on V_c).*

Proof. It is clear that $\bigcap_I K_i = \mathscr{C}$. The uniformity follows from the following version of Dini's lemma:

Lemma. *Let X and Y be compact topological spaces. Let K_i be a decreasing family of upper semicontinuous correspondences $K_i\colon X \to Y$. If*

$$\mathscr{C} = \bigcap_I K_i \mathscr{C}\colon X \to Y$$

is continuous then $K_i \downarrow \mathscr{C}$ uniformly.

Proof. Let U be an open entourage in $Y \times Y$ (see Kelley [7]) and let $X_i = \{x \mid K(x) \cap (\mid \mathscr{C}(x)|_U)^c \neq \emptyset\}$. From the continuity of $\mathscr{C}$ and the upper semicontinuity of K_i it follows that X_i is closed. Since $\bigcap_I X_i = \emptyset$ it follows that $\exists i\ X_i = \emptyset$. Since K_i is decreasing $X_j = \emptyset$, $\forall_j \geqslant i$, hence $K_i \downarrow \mathscr{C}$ uniformly. The classical formulation of Dini's lemma is a corollary, as is easily seen. Let $f_i\colon X \to \mathbb{R}$ be a decreasing family of continuous functions on the compact space X. Let $f_i \downarrow 0$. To prove $f_i \to 0$ uniformly we put

$$K_i(x) = \{t \mid 0 \leqslant t \leqslant f_i(x)\}, \qquad \mathscr{C}(x) = \{0\}\ \forall x$$

and apply the lemma.

8. σ-Continuous Games

Definition 6. A game $(\Omega, \mathscr{A}, v)$ is *continuous at* Ω if for each increasing sequence

$$(A_n)\left(\bigcup_n A_n = \Omega\right) \lim_{n\to\infty} v(A_n) = v(\Omega).$$

Schmeidler proved the following theorem [11]:

Theorem 1. *An exact game $(\Omega, \mathscr{A}, v)$ is continuous at Ω if and only if every element $\mu \in \mathscr{C}(v)$ is σ-additive. Moreover if $\mathscr{A}$ is a σ-algebra then*

(i) $\exists \mu_0 \in \mathscr{C}(v)$ *such that A is a null set if and only if* $\mu_0(A) = 0$,

(ii) $\mathscr{C}(v) \subset L_1(\Omega, \mathscr{A}, \mu_0)$.

Remark 1. Schmeidler did not prove that μ_0 can be chosen lying in $\mathscr{C}(v)$. However, replacing $1/2^i$ by $1/m_n$ (page 307 in Dunford–Schwartz [3]) yields immediately the existence of $\mu_0 \in \mathscr{C}(v)$.

Remark 2. Schmeidler conjectured that an exact game $(\Omega, \mathscr{A}, v)$ continuous at Ω can be extended to an exact game continuous at $\Omega = (\Omega, \tilde{\mathscr{A}}, \tilde{v})$ where $\tilde{\mathscr{A}}$ is the σ-algebra generated by $\mathscr{A}$. The following example answers this conjecture negatively, even for convex game.

Let Ω be a non-finite totally disconnected compact space. Let Σ be the algebra consisting of open closed sets. Let $v(\Omega) = 1$, $v(A) = 0$ otherwise. Clearly v is convex and hence exact. Moreover, if A_n is increasing, $\cup A_n = \Omega$, then $\exists n_0$ with $A_{n_0} = \Omega$, so clearly v is also continuous at Ω. Let $\tilde{v}$ be an exact extension of v. Let $\mu_0 \in \mathscr{C}(\tilde{v})$ such as in Schmeidler's theorem. The image of Σ in $\tilde{\Sigma}/\mu_0$ is dense ($\tilde{\Sigma}/\mu_0$ denotes the metric Boolean algebra associated with μ_0 [10]). Let $A \in \tilde{\Sigma}$ such that $\mu_0(\Omega \triangle A) = \mu_0(A^c) > 0$, then $\exists A_n \in \Sigma \mu_0(A_n \triangle A) \to 0$. $\forall n \, \exists \mu_n \in \mathscr{C}(\tilde{v})$ such that $\mu_n(A_n) = v(A_n)$. But $\exists n_0$, $\forall n \geqslant n_0$, $\mu_0(A_n{}^c) > 0$, and hence $\mu_n(A_n) = 0$. $\forall n \geqslant n_0$. Since $\mathscr{C}(\tilde{v})$ is $\sigma(L^1(\mu_0), L^\infty(\mu_0))$ weakly compact there is a convergent subsequence $\mu_n \to \mu$ (see [3]). Since convergence in μ measure is uniformly on weakly compact sets we obtain $\mu(A) = \lim \mu_n(A_n) = 0$. But $\mu \in \mathscr{C}(\tilde{v})$ and hence $\tilde{v}(A) \leqslant \mu(A) = 0$. It follows that $\tilde{v}(A) = 0$ if $\mu_0(A^c) > 0$ and $\tilde{v}(A) = 1$ if $\mu_0(A^c) = 0$. Hence

$$\mathscr{C}(\tilde{v}) = \left\{ f \in L^1(\Omega, \tilde{\Sigma}, \mu_0) \mid f \geqslant 0 \int_\Omega f \, d\mu = 1 \right\}.$$

Since $\mathscr{C}(\tilde{v})$ is $\sigma(L^1, L^\infty)$ compact it follows that the unit ball of $L^1(\mu_0)$ is weakly compact and hence $L^1(\mu_0)$ is finite dimensional. From the definition of Σ and $\tilde{\Sigma}$ it follows that μ_0 is a Baire measure and hence is a Radon measure on Ω.

It follows that μ_0 has a support consisting only of a finite number of points $\omega_1 \cdots \omega_n$. Let A be an open closed set not containing any of the points ω_j. Clearly $\mu_0(A) = 0$ and hence $\forall \mu \in \mathscr{C}(\tilde{v})$, $\mu(A) = 0$. It follows that $\max\{\mu(A) \mid \mu \in \mathscr{C}(\tilde{v})\} = 0$. But this is a contradiction to the exactness of $\tilde{v}$. Indeed $\tilde{v}(A^c) = 0$ since $A^c \neq \Omega$ and hence

$$\{\mu(A) \mid \mu \in \mathscr{C}(\tilde{v})\} = v(\Omega) - v(A^c) = 1.$$

The above counterexample is based on the following: Let Σ be an algebra and H a uniformly σ-additive family of measures, then the family $\tilde{H}$ of σ-additive extensions to $\tilde{\Sigma}$ is not necessarily uniformly σ-additive. It also follows that, if (Ω, Σ, v) is an exact game such that there is a subset K of $\mathscr{C}(v)$ satisfying

(i) $\forall A \in \Sigma \colon \exists \mu_n \in K \quad \mu_n(A) \to v(A)$,

(ii) the family $\tilde{K}$ of σ-additive extensions is relatively weakly compact (i.e., $\tilde{K}$ is uniformly σ-additive), then $\tilde{v}(\tilde{A}) = \inf\{\tilde{\mu}(\tilde{A}) \mid \tilde{\mu} \in K\}$ defines an exact extension $\tilde{v}$ of v.

If $\mathscr{A}$ is a σ-algebra, then $V_\sigma(\Omega, \mathscr{A})$ denotes the set of all σ-continuous convex games. The above counterexample shows that for σ-continuous games it is natural to suppose that $\mathscr{A}$ is a σ-algebra.

9. Weak Selection Theorems for σ-Continuous Convex Games

Selection Theorem 1. *Let X be a locally separable metric space. Let $g \colon X \to V_\sigma(\Omega, \mathscr{A})$ be a continuous function. Suppose that there is an open covering $(0_i)_{i \in I}$ of X such that $\forall i \in I \bigcup_{x \in 0_i} \mathscr{C}(g(x))$ is $\sigma(ca, ca')$ relatively compact (or, what is the same, $\sigma(ca, B)$ relatively compact). Then there is a function*

$$f \colon X \to ca(\Omega, \mathscr{A})$$

$\sigma(ca, ca')$ continuous such that $\forall x \in X$, $f(x) \in \mathscr{C}(g(x))$.

Proof. We know that a metric space is paracompact (see Kelley [7]), hence it is sufficient to prove the existence of a selection on each 0_i. Afterwards these selections are glued together by means of a locally finite partition of unity subordinate to the covering $(O_i)_I$. The existence of a selection on each 0_i follows immediately from the result of Amir and Lindenstrauss [1].

Corollary. *Under the assumptions of the theorem we can prove that for each closed set Y of X and each selection $f_Y \colon Y \to ca(\Omega, \mathscr{A})$ there is at least one extension of f_Y to a selection $f \colon X \to ca(\Omega, \mathscr{A})$.*

Proof. This follows immediately from the theorem and from a remark in [9] that the existence and the extension problems are equivalent.

Selection Theorem 2. *Let X be a topologic space which is the inductive limit of an increasing sequence of metrizable compact subsets X_n such that $\bigcup X_n = X$ (e.g., a hemicompact k-space). Then if $g: X \to V_\sigma$ is a continuous function there is a weakly continuous selection $f: X \to ca(\Omega, \mathscr{A})$, $f(x) \in \mathscr{C}(g(x))$, $\forall x \in X$.*

Proof. By Selection Theorem 1 there is for X_1 a selection f_1. By the corollary we can extend f_1 to a selection f_2 on X_2. By induction we define a selection $f: X \to ca(\Omega, \mathscr{A})$. On each X_n, f is continuous and hence by the inductive limit assumption f is continuous.

Selection Theorem 3. *Let $\mathscr{A}$ be a σ-algebra such that for all $\mu \in ca(\Omega, \mathscr{A})$ $\mathscr{A}/\mu$ is a separable metric space (this is, for instance, true if $\mathscr{A}$ is generated by a sequence of subsets of Ω). Let X be a paracompact space such that $L = \bigcup_{x \in X} \mathscr{C}(g(x))$ is $\sigma(ca, ca')$ relatively compact. Then there is a weakly continuous selection $f: X \to ca(\Omega, \mathscr{A})$ of $\mathscr{C} \circ g: X \to ca(\Omega, \mathscr{A})$.*

Proof. Since L is relatively weakly compact, its closed convex hull $\tilde{L}$ is weakly compact and lies in an $L_1(\Omega, \mathscr{A}, \mu)$ subspace of $ca(\Omega, \mathscr{A})$ [3]. Since $L_1(\Omega, \mathscr{A}, \mu)$ is separable, $\tilde{L}$ is metrizable and hence the selection follows from [9] and the remark that a metric convex compact set of a locally convex space is affinely homeomorphic with a compact convex set of a locally convex Fréchet space.

10. Approximation

Proposition 8. *If $(\mathscr{A}_j)_{j \in I}$ is a directed family of subtribes of the tribe $\mathscr{A}$ and if $\mathscr{A}$ is generated as σ-algebra by $\bigcup_I \mathscr{A}_i$, then K_i decreases uniformly to $\mathscr{C}$ on compact sets of $V_\sigma(\Omega, \mathscr{A})$.*

Proof. This follows immediately from Dini's lemma.

Let L be a subset of $V_\sigma(\Omega, \mathscr{A})$ such that $\bigcup_{v \in L} \mathscr{C}(v)$ is contained in an $L_1(\mu)$ subspace of $ca(\Omega, \mathscr{A})$ (e.g., let L be σ-compact). For each $\mathscr{A}_i$ we define ($(\mathscr{A}_i)_{i \in I}$ is a family of subtribes as in Proposition 6):

$$\tilde{K}_i: L \to L^1(\Omega, \mathscr{A}, \mu)$$

$$v \to \left\{ f \mid f \in L^1(\Omega, \mathscr{A}_i, \mu)\ \forall A \in \mathscr{A}_i: \int_A f\, d\mu\ v(A); \int_\Omega f\, d\mu = v(\Omega) \right\}.$$

By the projection theorem $\tilde{K}_i$ is nothing else then $E[\mathscr{C}(v) \mid \mathscr{A}_i]$ where $E[\ \mid \mathscr{A}_i]: L^1(\Omega, \mathscr{A}, \mu) \to L^1(\Omega, \mathscr{A}_i, \mu)$ denotes the conditional expectation operator (see Neveu [10]). It follows from Proposition 6 that for each compact subset X of L and each $\sigma(L^1, L^\infty)$ open set 0 $\exists i_0$ such that $\forall i \geqslant i_0$; $\forall v \in X$;

$\tilde{K}_i(v) \subset \mathscr{C}(v) + 0$. By an easy martingale argument (see Helms [5]) we have that $\forall v \in L$, $\forall f \in \mathscr{C}(v)$, $\exists f_i \in \tilde{K}_i(v)$, $f_i \to f$ in the strong $L^1(\mu)$ topology (observe that $E[f \mid \mathscr{A}_i] \to^{L^1} f$ and $E[f \mid \mathscr{A}_i] \in \tilde{K}_i(v)$).

We now prove the analog of Proposition 7 for the correspondences $\tilde{K}_i$.

PROPOSITION 9. *Let $\mathscr{A}_i$ be as in Proposition* 6. *Let L be a subset of $V_\sigma(\Omega, \mathscr{A})$ such that $\bigcup_{v \in L} \mathscr{C}(v)$ is contained in $L^1(\Omega, \mathscr{A}, \mu)$. Define K_i as above then $K_i \to \mathscr{C}(v)$ uniformly on compact sets in the weak topology $\sigma(L^1, L^\infty)$.*

Proof. We only have to prove the theorem for compact subsets and without loss of generality we may suppose L to be compact. The proposition then follows from the following martingale theorem (we are sure that this theorem is well known but we could not find any reference).

LEMMA. *Let $(f_i^\alpha)_{\alpha \in H}$ be a family of martingales such that $\mathscr{A}$ is generated by $\bigcup_I \mathscr{A}_i$, then $f_i^\alpha \to f^\alpha$ uniformly $\sigma(L^1, L^\infty)$ if the family of limits $\{f^\alpha \mid \alpha \in H\}$ is relatively $\sigma(L^1, L^\infty)$ compact.*

Proof. Suppose $\{f^\alpha \mid \alpha \in H\}$ is relatively $\sigma(L^1, L^\infty)$ compact, then by the usual criterion of weak compactness $\{f^\alpha \mid \alpha \in H\}$ is uniformly integrable.

Let $g_1 \cdots g_n$ be elements of L^∞ and suppose for simplicity that $\| g_j \|_\infty \leqslant 1$. We have to prove that

$$\lim_{i \in I} \sup_{\substack{\alpha \in H \\ j \leqslant n}} |\langle f^\alpha, g_j \rangle - \langle f_i^\alpha, g_i \rangle| = 0.$$

Let $\epsilon > 0$ and let K be sufficiently large such that $\| f^\alpha - f^\alpha \wedge K \|_{L^1} < \epsilon$ (such a K exists since $\{f^\alpha \mid \alpha \in H\}$ is uniformly integrable; $f^\alpha \wedge K$ denotes f^α truncated at K, i.e., $f^\alpha \wedge K = f^\alpha \cdot 1_{|f^\alpha| \leqslant K}$). Let i_0 be so large that $\forall i \geqslant i_0$, $\forall j \leqslant n$, we have

$$\| g_j - E[g_j \mid \mathscr{A}_i] \|_1 \leqslant \epsilon / K.$$

Now $\forall \alpha \in H$, $\forall i \geqslant i_0$,

$$\begin{aligned}
&|\langle f^\alpha, g_j \rangle - \langle E[f^\alpha \mid \mathscr{A}_i], g_j \rangle \\
&\quad \leqslant |\langle f^\alpha, g_j \rangle - \langle f^\alpha \wedge K, g_j \rangle| \\
&\qquad + |\langle f^\alpha \wedge K, g_j \rangle - \langle E[f^\alpha \wedge K \mid \mathscr{A}_i], g_j \rangle| \\
&\qquad + |\langle E[f^\alpha \wedge K \mid \mathscr{A}_i], g_j \rangle - \langle E[f^\alpha \wedge K \mid \mathscr{A}_i], E[g_j \mid \mathscr{A}_i] \rangle| \\
&\qquad + |\langle E[f^\alpha \wedge K \mid \mathscr{A}_i], E[g_j \mid \mathscr{A}_i] \rangle - \langle E[f^\alpha \mid \mathscr{A}_i], E[g_j \mid \mathscr{A}_i] \rangle| \\
&\qquad + |\langle E[f^\alpha \mid \mathscr{A}_i], E[g_j \mid \mathscr{A}_i] \rangle - \langle E[f^\alpha \mid \mathscr{A}_i], g_j \rangle| \\
&\leqslant \| f^\alpha - f^\alpha \wedge K \|_1 \| g_j \|_\infty + K \cdot \| g_j - E[g_j \mid \mathscr{A}_i] \|_1 \\
&\qquad + 0 + \| E[f^\alpha \wedge K - f^\alpha \mid \mathscr{A}_i] \|_1 \cdot \| E[g_j \mid \mathscr{A}_i] \|_\infty \\
&\qquad + 0 \\
&\leqslant \epsilon + K \cdot \epsilon / K + \epsilon = 3\epsilon
\end{aligned}$$

(since $\| E[g_j \mid \mathscr{A}_i] \|_\infty \leqslant \| g_j \|_\infty$ and $\| E[h \mid \mathscr{A}_i] \|_1 \leqslant \| h \|_1$).

11. Exposed Points of Cores of σ-Continuous Convex Games

In Section 4, we gave a characterization of all extreme points of finite games. For arbitrary games this characterization becomes somewhat more difficult since an extreme point of an infinite dimensional compact convex set is not necessarily a support point for a hyperplane. However for σ-continuous convex games the core is a weakly compact convex subset of a Banach space and so we can apply the results of [1], [2], and [8].

Definition 7. If B is a Banach space and $K \subset B$ is weakly compact convex then $x \in K$ is called *exposed* if $\exists x^* \in B^*$ such that

$$\langle x^*, x\rangle > \langle x^*, y\rangle \qquad \text{for all } y \in K, y \neq x.$$

A profound theorem due to Amir, Corson, and Lindenstrauss [1, 2, 8] states that a weakly compact convex subset of a Banach space is the closed convex hull of its exposed points (by the inverse Krein–Milman theorem it then follows that the exposed points are dense in the extreme points).

In the following proposition $v \in V_\sigma$, and λ is a measure such that $\mathscr{C}(v) \subset L^1(\Omega, \mathscr{A}, \lambda)$.

Proposition 10. *If $f \in \mathscr{C}(v)$ is an exposed point then there is a mapping C: $[01] \to \mathscr{A}$ $t \to C_t$ such that*

(i) $C(0) = \Omega$,

(ii) $C_t \subset C_s$ *for all* $s \leqslant t$,

(iii) $\bigcap_{t<s} C_t = C_s$,

(iv) $v(C_t) = \int_{C_t} f\, d\lambda$ *for all* t.

Proof. Let $g \in L^\infty(\Omega, \mathscr{A}, \lambda)$ be the supporting hyperplane. We may suppose $0 \leqslant g \leqslant 1$ (eventually we add a constant to g). Let

$$C_t = \{\omega \mid g(\omega) \geqslant t\}.$$

Clearly C satisfies (i), (ii), (iii). We now define

$$g_n = \sum_{j=2}^{n} \frac{1}{2^n} l_{C_{j/2^n}}, \qquad j = 0, 1,..., 2^n.$$

Then $g_n \to g$ in the L^∞ norm. Let

$$K_n = \{f \mid f \in \mathscr{C}(v) \text{ with } \langle g_n, f\rangle = \min\langle g_n, \mathscr{C}(v)\rangle\}.$$

We have that $f_n \in K_n$ implies $f_n \to f$ weakly. (Indeed $s\colon L^\infty \to \mathbb{R}$; $h \to \min\langle h, \mathscr{C}(v)\rangle$ is continuous and hence

$$\langle g_n, f_n\rangle = \min\langle g_n, \mathscr{C}(v)\rangle \to \min\langle g, \mathscr{C}(v)\rangle = \langle g, f\rangle.)$$

Since f_n has an adherent point by weak compactness and since f can be the only adherent point to f_n, $f_n \to f$ weakly. But for f_n we have that

$$\int_{C_{j/2^n}} f_n \, d\lambda = v(C_{t/2^n}), \qquad \forall t = 0,\ldots, 2^n.$$

Hence $\int_{C_t} f\, d\lambda = v(C_t)$ for all t dyadic in [01]. We now observe that by convexity $v\colon \mathscr{A}/\lambda \to \mathbb{R}_+$ is a continuous function and hence by the density of the diadic numbers $\int_{C_t} f\, d\lambda = v(C_t)$ for all $t \in [01]$.

Before proving a partial converse to Proposition 10 we first give a method to find $\min\langle g, \mathscr{C}(v)\rangle$ where $0 \leqslant g \leqslant 1$. Let g_n be as in the proof of Proposition 10. Clearly $\exists f_n$, $\min\langle g_n, \mathscr{C}(v)\rangle = \langle g_n, f_n\rangle$, and hence

$$\int_{C_{j/2^n}} f_n \, d\lambda = v(C_{j/2^n})$$

for all $j = 1,\ldots, 2^n$. Clearly f_n has a convergent subsequence $f_n \to f$ and $\langle g, f\rangle = \min\langle g, \mathscr{C}(v)\rangle$. From the same proof it follows

$$\int_{C_t} f\, d\lambda = v(C_t)$$

for all $t \in [01]$. Hence $\min\langle g, \mathscr{C}(v)\rangle = \int t\, dV_t$ where V_t is defined as

$$V_t = \int_{\{\omega \mid g(\omega) < t\}} f\, d\lambda = v(\Omega) - v(C_t) = \bar{v}(C_t{}^c).$$

Let now f' be a second function such that $\int g \cdot f'\, d\lambda = \min\langle g, \mathscr{C}(v)\rangle$, then $\langle g, f'\rangle = \int t\, dV_t'$ where

$$V_t' = \int_{\{\omega \mid g(\omega) < t\}} f'\, d\lambda \leqslant \bar{v}(C_t{}^c) = V_t$$

since $f' \in \mathscr{C}(v)$. However, $\int t\, d(V_t - V_t') = 0$, and hence $V_t = V_t'$, $\forall t \in [01]$. This proves that $\forall f \langle g, f\rangle = \min\langle g, \mathscr{C}(v)\rangle$ one has

$$\int_{C_t} f\, d\lambda = v(C_t).$$

PROPOSITION 11. *If* $C\colon [01] \to \mathscr{A}$ *is a function satisfying* (i), (ii), (iii) *of Proposition* 10, *then* $\exists f \in \mathscr{C}(v)$ *such that* $\int_{C_t} f \, d\lambda = v(C_t)$, $\forall t \in [01]$. *If, moreover,* $\mathscr{A}$ *is a subtribe of the tribe generated by the null sets and the family* C_t, *then* f *is exposed.*

Proof. The existence of f follows from Corollary 3 of Lemma 2. From the remarks after Proposition 10 it follows that, if $\langle f', g\rangle = \min\langle g, \mathscr{C}(v)\rangle$, then $\int_{C_t} f' \, d\lambda = \int_{C_t} d\lambda = v(C_t)$ and hence $f' = f$. Hence

$$\langle g, f\rangle < \langle g, f'\rangle \; \forall f' \in \mathscr{C}(v), \qquad f' \neq f.$$

12. AN EXAMPLE

Consider the following game:

$$\Omega = [01], \qquad \mathscr{A} = \{A \mid A \text{ Borel subset of } [01]\},$$

$$v(A) = \frac{e^{|A|} - 1}{e - 1},$$

where $|A|$ denotes Lebesgue measure of A.

(1) V is convex and σ-continuous. The σ-continuity follows from the definition. The convexity follows from the inequality

$$e^a + e^b \leqslant 1 + e^{a+b} \qquad \text{for all } a \geqslant 0, \quad b \geqslant 0.$$

(2) $\bar{v}(A) = v(\Omega) - v(A^c) = 1 - \dfrac{e^{|A^c|} - 1}{e - 1} = \dfrac{e - e^{|A^c|}}{e - 1} = e^{|A^c|} v(A).$

(3) A function $f \in L^1[01]$ is an exposed point of $\mathscr{C}(v) \subset L^1[01]$ if and only if f is of the form $f(x) = (e^{\theta(x)})/(e - 1)$, where $\theta\colon [01] \to [01]$ is a measure preserving measurable bijection. This follows immediately from Propositions 10 and 11 (in particular one has

$$\int_A f(x) \, dx \geqslant \frac{e^{|A|} - 1}{e - 1}$$

for all Borel sets. The reader should try to prove this inequality directly, i.e., without using convex games).

(4) $1 \in \mathscr{C}(v)$. Indeed

$$|A| \geqslant \frac{e^{|A|} - 1}{e - 1} \qquad \text{for all } A \in \mathscr{A}.$$

(5) 1 is a point of symmetry of $\mathscr{C}(v)$, i.e., if $f \in \mathscr{C}(v)$ then $2 - f \in \mathscr{C}(v)$. Indeed

$$\forall A \int_A (2 - f)\, d\lambda \geqslant 2 \mid A \mid - \bar{v}(A) = 2 \mid A \mid \frac{e - e^{|A^c|}}{e - 1} \geqslant v(A).$$

(6) $\forall f \in \mathscr{C}(v)\colon \dfrac{1}{e-1} \leqslant f \leqslant 2 - \dfrac{1}{e-1} = \dfrac{2e-3}{e-1}$.

(7) The core $\mathscr{C}(v)$ is $\sigma(L^1, L^\infty)$ compact but not L^1 compact and the set of extreme points of $\mathscr{C}(v)$ is not closed. Indeed let θ: $[01] \rightarrow [01]$ be a *strongly mixing operator* [10] and let f be an exposed point, $f \circ \theta^n = f_n$ is a sequence of exposed points converging to 1 in the $\sigma(L^1, L^\infty)$ topology. From this it follows that the set of exposed points is not closed. That $\mathscr{C}(v)$ is not L^1 compact follows from the observation that no subsequence of (f_n) converges to 1 in the L^1 norm.

References

1. D. Amir and J. Lindenstrauss, The structure of weakly compact sets in Banach spaces, *Ann. of Math.* **88** (1968), 35–46.
2. H. H. Corson and J. Lindenstrauss, On weakly compact subsets of Banach spaces, *Proc. Amer. Math. Soc.* **171** (1966), 407–412.
3. N. Dunford and J. T. Schwartz, "Linear Operators," Part I, Interscience, New York, 1963.
4. Ky Fan, On systems of linear inequalities, *Ann. of Math. Stud.* **38** (1956), 99–156.
5. L. L. Helms, Mean convergence of martingales, *Trans. Amer. Math. Soc.* **87** (1958), 439–446.
6. Y. Kannai, Countably additive measures in cores of games, *J. Math. Anal. Appl.* **27** (1969), 227–240.
7. J. L. Kelley, "General Topology," Van Nostrand, Princeton, 1955.
8. J. Lindenstrauss, On operators which attain their norm, *Israel J. Math.* **1** (1963), 139–148.
9. E. Michael, Continuous selections, I, *Ann. of Math.* **63** (1956), 361–382.
10. J. Neveu, "Bases Mathématiques du Calcul des Probabilités," Masson, Paris, 1964.
11. D. Schmeidler, Cores of exact games, I, University of California, Berkeley, Center for Research in Management Science, 1971.
12. D. Schmeidler, On balanced games with infinitely many players, Research Program in Game Theory and Mathematical Economics, RM 28, The Hebrew University of Jerusalem.
13. L. Shapley, Notes on n-person games, VII. Cores of convex games, The Rand Corporation, Santa Monica, 1965.
14. L. Shapley, Cores of convex games, *Internat. J. Game Theory* **1**(1), 1971.
15. D. Sondermann, Existence of stable profit distributions and equilibria in coalition production economies, CORE Discussion Paper 7141, Center for Operation Research and Econometrics, Leuven, Belgium.